

Planck mass and Dilaton field as a function of the noncommutative parameter

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Abstract

A deformed Bianchi type I metric in noncommutative gauge gravity is obtained. The gauge potential (tetrad fields) and scalar curvature are determined up to the second order in the noncommutativity parameters. The noncommutativity correction to the Einstein-Hilbert action is deduced. We obtain the Planck mass, on noncommutative space-time as a function of the noncommutative parameter θ , which implies that noncommutativity has modified the structure and topology of the space-time.

Keywords: *Noncommutative field theory, Gauge field theory, Quantum gravity, Quantum fields in curved spacetime.*

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1 Introduction

Matter and field are basic concepts of classical field theories. They play a fundamental role in the general relativity theory [1], where the Einstein tensor G_μ^ν is expressed in terms of the geometry of space-time, and the matter is represented by its momentum energy density tensor T_μ^ν . These two intrinsic concepts are connected by the Einstein field equation

$$G_\mu^\nu = -8\pi T_\mu^\nu. \quad (1)$$

According to eq. (1), a given distribution of matter (sources) determines the geometric properties of space-time. One can regard this as the creation of space-time geometry by matter. Now, one can read eq. (1) in the opposite direction, and expect the creation of matter by geometry, which is an interesting mechanism for creating particles.

The standard concept of space time as a geometric manifold is based on the notion of points being locally labelled by a finite number of real coordinates. However, it is generally believed that this picture of space time as a manifold should break down at very short distances of the order of the Planck length. This implies that the mathematical concepts of high energy physics have to be changed or more precisely our classical geometry concepts may not be well suited for the description of physical phenomenon at short distances.

Noncommutativity is a mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables such as position and momentum. It would be interesting to look for a mechanism of creating particles in the noncommutative geometry. There are several motivations to speculate that space-time becomes non-commutative at very short distances when quantum gravity becomes relevant. Moreover, in string theories, the non-commutative gauge theory appears as a certain limit in the presence of a background field [2 – 3]. In this context, a gauge field theory with star products and Seiberg-Witten maps is used .

We shall work with the noncommutative canonical space-time, which can be realized by the generalisation of the commutation relation for the canonical variables (coordinate-momentum operators) to non-trivial ones of coordinate operators. It was shown that combining Heisenberg uncertainty principle with Einstein theory of classical gravity leads to the conclusion that ordinary space-time loses any operational meaning at short distances, that is a space-time coordinate with a great accuracy ϵ causes an uncertainty in momentum or energy of the order of ϵ^{-1} which is transmitted to the system and concentrated at some time in the localization point . Exploring the limitations of localisation measurements which are due to the possible black hole creation by concentration of energy, one arrives at uncertainty relations among space-time coordinates which can be traced back to the noncommutative commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu]_* = i\theta^{\mu\nu}, \quad (2)$$

where \hat{x} are the coordinate operators and $\theta^{\mu\nu}$ is the noncommutativity parameter and is of dimension of length-squared, and corresponds to the smallest patch of area in physical space one may “observe”, similar to the role \hbar plays in $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, defining the corresponding smallest patch of phase space in quantum mechanics.

It is a challenge to formulate a theory of gravitation on a non-commutative manifold. The main problem lies in the fact that it is difficult to implement symmetries such as general coordinate covariance and local Lorentz invariance and to define derivatives which are torsion-free and satisfy the metricity condition. In a flat space-time, to get noncommutative local gauge theories with Lorentz violation symmetry, a formulation within the enveloping algebra approach has been proposed [2] – [4]. Following a similar path, a gauge formulation of gravity is proposed [3 – 5 – 6]. It is a theory of general relativity on curved spacetime with canonical preservation of noncommutative spacetime commutation relations and based partially on implementing symmetries on flat noncommutative spacetime. The non-locality in time-like noncommutativity θ^{0i} ($i = 1, 2, 3$) leads to the unitarity violation [7]. To overcome this problem we obtain the generalised local Lorentz and general coordinates infinitesimal transformations of the noncommutative space-time, which are exact symmetries of the canonical non-commutative spacetime commutation relations without any further constraints like unimodularity [8].

In this paper, we present a deformed Bianchi type I metric in non-commutative gauge theory of gravitation. Then using the Seiberg-Witten map we calculate

the non-commutative corrections for the gauge potentials (tetrad fields) up to the second order of the expansion in θ . Using the results we determine the deformed Bianchi type I metric. The correction is obtained up to the second order of the noncommutativity parameter, and we compute the Ricci scalar curvature in a noncommutative cosmological anisotropic Bianchi I universe. The noncommutative correction to the Einstein-Hilbert action is deduced.

2 The gauge theory

The anisotropic Bianchi type I space-time metric is expressed by the line-element:

$$ds^2 = -dt^2 + t^2 (dx^2 + dy^2) + dz^2. \quad (3)$$

The corresponding metric $g_{\mu\nu}$ has the following non-zero components:

$$g_{00} = -1, g_{11} = t^2 = g_{22}, g_{33} = 1. \quad (4)$$

The tetrads $e_\mu^a(x)$, $a = 0, 1, 2, 3$, and the spin connections $\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x)$, $[ab] = [01], [02], [03], [12], [13], [23]$, are written as follows:

$$\omega_\mu^{ab} = e_\rho^a e^{\nu b} \Gamma_{\mu\nu}^\rho - e^{\nu b} \partial_\mu e_\nu^a. \quad (5)$$

Here $e_a^\nu(x)$ denotes the inverse of $e_\mu^a(x)$ satisfying the usual properties:

$$e_\mu^a e_b^\mu = \delta_b^a, e_\mu^a e_a^\nu = \delta_\mu^\nu, \quad (6)$$

and $\Gamma_{\mu\nu}^\rho$ is the affine connection, which is written in function of the metric $g_{\mu\nu}$ as:

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \quad (7)$$

Then, the corresponding components of the strength tensor can be written in the standard form as the torsion tensor:

$$R_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + (\omega_\mu^{ab} e_\nu^d + \omega_\nu^{ab} e_\mu^d) \eta_{bd}, \quad (8)$$

with η_{bd} the flat space metric, and the curvature tensor:

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + (\omega_\mu^{ac} \omega_\nu^{db} + \omega_\nu^{ac} \omega_\mu^{db}) \eta_{cd}. \quad (9)$$

The particular form of the line-element gravitational gauge field is given by the following ansatz:

$$e_\mu^0 = (1, 0, 0, 0), e_\mu^1 = (0, t, 0, 0), e_\mu^2 = (0, 0, t, 0), e_\mu^3 = (0, 0, 0, 1), \quad (10)$$

and

$$\Gamma_{11}^0 = \Gamma_{22}^0 = t, \Gamma_{10\nu}^1 = \Gamma_{20}^2 = \frac{1}{t}, \omega_1^{01} = \omega_2^{02} = 1. \quad (11)$$

Here we only give the expressions of $R_{\mu\nu}^{ab}$ components, which we need to use further in the derivation of the expressions of tetrads:

$$R_{12}^{12} = R_{21}^{21} = R_{21}^{12} = -R_{12}^{21} = 1. \quad (12)$$

The scalar curvature is $R = R_{\mu\nu}^{ab} e_\mu^a e_b^\nu = 2/t^2$.

3 Noncommutative gauge theory

We assume that the noncommutative canonical structure of space time is defined by eq. (1) where $*$ is the star product defined between the functions f and g over this spacetime:

$$(f * g)(x) = f(x) \exp \left(\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x). \quad (13)$$

The gauge field for the noncommutative canonical space-time are denoted by $\hat{\omega}_\mu^{ab}(x, \theta)$, subject to the conditions[9, 10]:

$$\hat{\omega}_\mu^{ab+}(x, \theta) = -\hat{\omega}_\mu^{ba}(x, \theta) = \hat{\omega}_\mu^{ab}(x, -\theta) = \hat{\omega}_\mu^{abr}(x, \theta). \quad (14)$$

By expanding the gauge fields in powers of θ^2 ,

$$\hat{\omega}_\mu^{ab}(x, \theta) = \omega_\mu^{ab}(x) + \theta^{\alpha\beta} \omega_{\mu\alpha\beta}^{ab}(x) + \theta^{\alpha\beta} \theta^{\gamma\delta} \omega_{\mu\alpha\beta\gamma\delta}^{ab}(x) + O(\theta^3), \quad (15)$$

the above conditions then imply the following:

$$\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x), \omega_{\mu\alpha\beta}^{ab}(x) = \omega_{\mu\alpha\beta}^{ba}(x). \quad (16)$$

Using the Seiberg-Witten map [2], one obtains the following noncommutative corrections up to the second order [11]:

$$\omega_{\mu\alpha\beta}^{ab}(x) = -\frac{i}{4} \{ \omega_\alpha, \partial_\beta \omega_\mu + R_{\beta\mu} \}^{ab}, \quad (17)$$

$$\begin{aligned} \omega_{\mu\alpha\beta\gamma\delta}^{ab}(x) = & \frac{1}{32} \left([\omega_\gamma, 2 \{ R_{\delta\alpha}, R_{\mu\beta} \} - \{ \omega_\alpha, (D_\beta R_{\delta\mu} + \partial_\beta R_{\delta\mu}) \} - \right. \\ & - \partial_\delta \{ \omega_\alpha, (\partial_\beta \omega_\mu + R_{\beta\mu}) \}]^{ab} + [\partial_\alpha \omega_\gamma, \partial_\beta (\partial_\delta \omega_\mu + R_{\delta\mu})]^{ab} - \\ & \left. - \{ \{ \omega_\alpha, (\partial_\beta \omega_\gamma + R_{\beta\gamma}) \}, (\partial_\delta \omega_\mu + R_{\delta\mu}) \}^{ab} \right), \end{aligned} \quad (18)$$

where

$$\{A, B\}^{ab} = A^{ac} B_c^b + B^{ac} A_c^b, [A, B]^{ab} = A^{ac} B_c^b - B^{ac} A_c^b, \quad (19)$$

and

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - \omega_\nu^{ac} \omega_\mu^{cb}, D_\mu R_{\alpha\beta}^{ab} = \partial_\mu R_{\alpha\beta}^{ab} + (\omega_\mu^{ac} R_{\alpha\beta}^{db} + \omega_\mu^{bc} R_{\alpha\beta}^{da}) \eta_{cd}. \quad (20)$$

The result for \hat{e}_μ^a up to the second order in θ is:

$$\begin{aligned} \hat{e}_\mu^a = & e_\mu^a - \frac{i}{4} \theta^{\alpha\beta} (\omega_\alpha^{ac} \partial_\beta e_\mu^c + (\partial_\beta \omega_\mu^{ac} + R_{\beta\mu}^{ac}) e_\mu^c) \\ & + \frac{1}{32} \theta^{\alpha\beta} \theta^{\gamma\delta} (2 \{ R_{\delta\alpha}, R_{\mu\beta} \}^{ac} e_\gamma^c - \omega_\gamma^{ac} (D_\beta R_{\delta\mu}^{cd} + \partial_\beta R_{\delta\mu}^{cd}) e_\alpha^d \\ & - \{ \omega_\alpha (D_\beta R_{\delta\mu} + \partial_\beta R_{\delta\mu}) \}^{ad} e_\gamma^d - \partial_\delta \{ \omega_\alpha (\partial_\beta \omega_\mu + R_{\beta\mu}) \}^{ac} e_\gamma^c \\ & - \omega_\gamma^{ac} \partial_\delta \{ \omega_\alpha \partial_\beta e_\mu^d + (\partial_\beta \omega_\mu^{cd} + R_{\beta\mu}^{cd}) e_\alpha^d \} + \partial_\alpha \omega_\gamma^{ac} \partial_\beta \partial_\delta e_\mu^c \\ & - \partial_\beta (\partial_\delta \omega_\mu^{ac} + R_{\delta\mu}^{ac}) \partial_\alpha e_\gamma^c - \{ \omega_\alpha (\partial_\beta \omega_\gamma + R_{\beta\gamma}) \}^{ac} \partial_\delta e_\mu^c \\ & - \partial_\beta (\partial_\delta \omega_\mu^{ac} + R_{\delta\mu}^{ac}) (\omega_\alpha^{cd} \partial_\beta e_\gamma^d + (\partial_\beta \omega_\gamma^{cd} + R_{\beta\gamma}^{cd}) e_\alpha^d) + \mathcal{O}(\theta^3). \end{aligned} \quad (21)$$

Then we can introduce a noncommutative metric by the formula:

$$\hat{g}_{\mu\nu} = \frac{1}{2} \eta_{ab} (\hat{e}_\mu^a * \hat{e}_\nu^{b+} + \hat{e}_\mu^b * \hat{e}_\nu^{a+}), \quad (22)$$

where the \hat{e}_μ^{a+} is the complex conjugate of the noncommutative tetrad fields given in (21).

3.1 Second order correction to Bianchi Type I

We choose the coordinate system so that the noncommutative parameters $\theta^{\alpha\beta}$ are given by [12] :

$$\theta^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & \theta \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \end{pmatrix}. \quad (23)$$

The nonzero components of the tetrad fields \hat{e}_μ^a are:

$$\hat{e}_1^1 = t \left(1 - \frac{3}{32} \theta^2 \right) + \mathcal{O}(\theta^3), \quad (24)$$

$$\hat{e}_2^1 = -i \frac{t}{4} \theta + \mathcal{O}(\theta^3), \quad (25)$$

$$\hat{e}_1^2 = i \frac{t}{4} \theta + \mathcal{O}(\theta^3), \quad (26)$$

$$\hat{e}_2^2 = t \left(1 - \frac{3}{32} \theta^2 \right) + \mathcal{O}(\theta^3), \quad (27)$$

$$\hat{e}_3^3 = 1 + \mathcal{O}(\theta^3), \quad (28)$$

$$\hat{e}_0^0 = 1 + \mathcal{O}(\theta^3). \quad (29)$$

Then, using the definition (22), we obtain the following non-zero components of the noncommutative metric $\hat{g}_{\mu\nu}$ up to the second order of θ :

$$\hat{g}_{11} = t^2 - \frac{3}{32} t^2 \theta^2 + \mathcal{O}(\theta^3), \quad (30)$$

$$\hat{g}_{22} = t^2 - \frac{3}{32} t^2 \theta^2 + \mathcal{O}(\theta^3), \quad (31)$$

$$\hat{g}_{33} = 1 + \mathcal{O}(\theta^3), \quad (32)$$

$$\hat{g}_{00} = -1 + \mathcal{O}(\theta^3), \quad (33)$$

such that for $\theta \rightarrow 0$ we obtain the ordinary ones. Then we can calculate the noncommutative correction to the Ricci scalar curvature given by:

$$\hat{R} = \hat{e}_a^\mu * \hat{R}_{\mu\nu}^{ab} * \hat{e}_b^\nu = \frac{2}{t^2} \left(1 - \frac{3}{32} \theta^2 \right) \approx R \exp \left(-\frac{3}{32} \theta^2 \right), \quad (34)$$

where $R = 2/t^2$ is the ordinary scalar curvature.

The Einstein-Hilbert action is:

$$S_g = -\frac{1}{16\pi G} \int d^4x R \sqrt{-g} \exp\left(-\frac{3}{16}\theta^2\right). \quad (35)$$

When the effective coupling $\exp\left(\frac{3}{16}\theta^2\right)$ is small, then the Dilaton field is proportional to θ^2 [13]. In this case the action (35) can be written in this form:

$$S_g = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\hat{R} + W(\theta^2) \right], \quad (36)$$

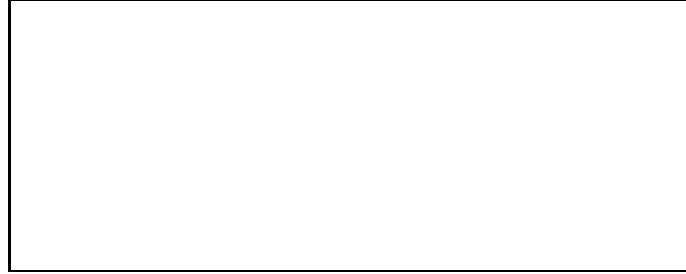
where the $W(\theta^2)$ is the Dilaton potential. We note through eq. (31) that the noncommutative geometry creates the Dilaton field which is automatically within the gravity and we think that this new coupling represents the source matter. The action (35) as such also undergoes a modification that is reflected by the coupling constant $\hat{k} = k \exp\left(\frac{3}{16}\theta^2\right)$ (k is the ordinary coupling). This expression is the same as that when the action is written in the space of five dimensions; then we can say that noncommutativity is responsible of the fifth dimension of the gravity. With all this the equation of motion derived from (35) is:

$$e^{-\frac{3}{16}\theta^2} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0. \quad (37)$$

The equation (37) takes the same form as in the ordinary space-time although it is multiplied by the factor $e^{-\frac{3}{16}\theta^2}$ which has no effect on it. However, this factor influences the coupling constant k as we have seen previously. Thus the Planck mass is a function of the parameter noncommutative as follows:

$$\hat{M}_p = M_p \left(1 - \frac{3}{32}\theta^2 \right) \sim M_p \exp\left(-\frac{3}{32}\theta^2\right), \quad (38)$$

where M_p is the ordinary Planck mass. To get an idea about the qualitative effect of noncommutativity, we display the ratio M_p/\hat{M}_p as a function of the noncommutative parameter θ (see Fig. below). Note that if the noncommutative parameter θ increases the ratio M_p/\hat{M}_p increases. These different values are essentially characterised by different the gravitational constant values on noncommutative space-time. This variation depends to the noncommutative parameter θ . Thus the Planck mass, in noncommutative space-time, is composed of two parts: the ordinary one and the second part which is negative and expressed in the second order of θ . We understand that this negative value represents the part of the matter which is kept secret in the noncommutative coordinates; which can explain the dark matter.



Plank mass ratio versus theta

4 Conclusions

Throughout this work we used Seiberg-Witten maps and the Moyal product up to the second order of the noncommutativity parameter θ , we choose the Bianchi I universe and derive both the second order noncommutative correction of the metric, and the deformed Einstein-Hilbert action. In addition we obtained the Planck mass on noncommutative space-time as a function of noncommutative parameter θ , which implies that the noncommutativity has modified the structure and topology of the space-time. This induces the connection between the mass and parameter responsible of noncommutative geometry which permits us to understand the dark matter and dark energy especially if we take the noncommutative parameter dependent on the coordinates.

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